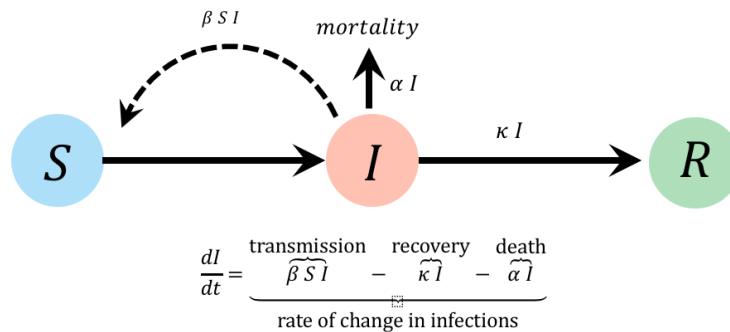**Variables**
S: susceptible
I: infected (infectious)
R: recovered (not infectious)

$$\boxed{\text{SIR Model}}$$

Parameters
β: transmission rate
κ: recovery rate
α: mortality rate ("virulence")

$\beta\, S\, I$ *mortality*

$\alpha\, I$

S ———→ I ——$\kappa\, I$——→ R

$$\frac{dI}{dt} = \overbrace{\beta\, S\, I}^{\text{transmission}} - \overbrace{\kappa\, I}^{\text{recovery}} - \overbrace{\alpha\, I}^{\text{death}}$$

$$\underbrace{\phantom{\frac{dI}{dt} = \beta S I - \kappa I - \alpha I}}_{\text{rate of change in infections}}$$

Question 2 [20 points per page]: The SIR model is a classic model for the spread of disease, such as COVID-19. Here, we track the proportion of a population that is susceptible (S), infected (I), or resistant (R), which sum to one, and use this model to obtain insights about disease spread.

(a) When the population was mainly susceptible (as was true in BC before Omicron), the number of susceptible individuals, S, could be treated as a constant (a fixed parameter), because infections were few relative to the total size of the population. Treating β, S, κ, α as fixed positive constants, the above equation $\frac{dI}{dt} = \beta\, S\, I - \alpha\, I - \kappa\, I$ can be written in the form $\frac{dI}{dt} = r\, I$.

[**Tick one from each column**] Which of the following describes the equation, $\frac{dI}{dt} = r\, I$?

TICK ONE OF:
X Equivalent to exponential growth model
☐ Equivalent to logistic model
☐ Equivalent to haploid selection model
☐ Equivalent to diploid selection model

TICK ONE OF:
X Differential equation
☐ Recursion equation
☐ Difference equation

TICK ONE OF:
X Continuous-time model
☐ Discrete-time model
☐ Not enough information given

(b) Fill in the following blanks, relating the original equation $\frac{dI}{dt} = \beta\, S\, I - \alpha\, I - \kappa\, I$ to $\frac{dI}{dt} = r\, I$. We can write $r = \underline{\ \beta\, S - \alpha - \kappa\ }$, in terms of the original parameters β, S, κ, α. For the number of infections to grow over time, we need $r\underline{\ >\ }0$ [**choose >, <, or =**]. This tells us that the disease will spread as long as the proportion of susceptibles in the population satisfies $S > \underline{\ (\alpha + \kappa)/\beta\ }$ [write as a function of the other parameters, β, κ, α].

(c) At this point in the pandemic, many people have had prior exposure to the virus, either through vaccination or past infection, building up immunity. This resistance wanes over time, however, at rate δ. The proportion of susceptible individuals is expected to change according to $\frac{dS}{dt} = \delta\, R - \beta\, S\, I$ with waning. **Add an arrow** describing this waning to the figure above; **write the total flow rate** by the arrow. If the flow occurs at a per capita rate, then multiply it by the appropriate variable. Add a solid arrow from R to S with $\delta\, R$ as the total flow rate over that arrow.

(d) In a population where most individuals are resistant ($R \sim 1$) and few are infected ($I \sim 0$), the above differential equation simplifies to $\frac{dS}{dt} = \delta$. You think that the general solution is $S(t) = \delta\, t + S(0)$, where S_0 is the proportion of susceptible individuals at time 0. Verify that this general solution satisfies the differential equation $\frac{dS}{dt} = \delta$ (only partial credit will be given for other checks).

Taking the derivative of $S(t) = \delta\, t + S_0$ with respect to t gives: $\frac{dS}{dt} = \frac{d(\delta\, t + S_0)}{dt} = \frac{d(\delta\, t)}{dt} + \frac{d(S_0)}{dt}$. Since S_0 is a constant in the model (the initial proportion of susceptibles), $\frac{d(S_0)}{dt} = 0$ and $\frac{dS}{dt} = \delta\frac{dt}{dt} = \delta$.

(e) Accounting for waning by using the solution in part (d), we expect the infected class to change over time according to $\frac{dI}{dt} = \beta\,(\delta\, t + S_0)\, I - \alpha\, I - \kappa\, I$. Begin a separation of variables by rewriting this differential equation with the variables separated.

$$\int \frac{1}{I}\, dI \;=\; \int \beta\,(\delta\, t + S_0) - \alpha - \kappa \quad dt$$

(f) Integrate both sides and solve for the proportion of infected individuals, $I(t)$, at time t, given an initial number I_0 at time 0.

$$\ln(I) = \frac{1}{2}\beta\,\delta\, t^2 + (\beta\, S_0 - \alpha - \kappa)\, t + constant$$

solving at time $t{=}0$ for the constant of integration: $\ln(I_0) = constant$ and exponentiation both sides:

$$I = \exp\left(\frac{1}{2}\beta\,\delta\, t^2 + (\beta\, S_0 - \alpha - \kappa)\, t + \ln(I_0)\right)$$

$$I = e^{\frac{1}{2}\beta\,\delta\, t^2}\, e^{(\beta\, S_0 - \alpha - \kappa)\, t}\, I_0$$

(g) Assuming that the population was initially so resistant that the disease could not spread, at what time, t, do you expect the disease to start spreading due to waning immunity? You can answer this question by determining the time at which $\frac{dI}{dt}$ becomes positive.

Note that $\frac{dI}{dt} = \beta\,(\delta\, t + S_0)\, I - \alpha\, I - \kappa\, I$ if S_0 is too small, then I will decline over time at $t{=}0$, because there are too few people to infect (no matter what the initial value of I). $\frac{dI}{dt}$ will then become positive when $\beta\,(\delta\, t + S_0)I - \alpha\, I - \kappa\, I > 0$, that is, $t = (\alpha + \kappa - \beta\, S_0)/(\delta\,\beta)$

(h) You use maxima to find the general solution to the equation, $\frac{dI}{dt} = \beta\,(\delta\,t + S_0)\,I - \alpha\,I - \kappa\,I$, but you have three errors in your code. Circle and correct the three errors. [δt needs a multiplication sign before the *t*; "exp_model" should be "sir_model", and $-\alpha\,I - \kappa\,I$ should not be written as $-(\alpha\,I - \kappa\,I)$ as this changes the sign of the last term.]

```
sir_model: 'diff(I, t) = β·(δt+S0)·I - (α·I - κ·I);
sir_sol: ode2(exp_model, I, t);
ic1(sir_sol, t = 0, I = I0);
```
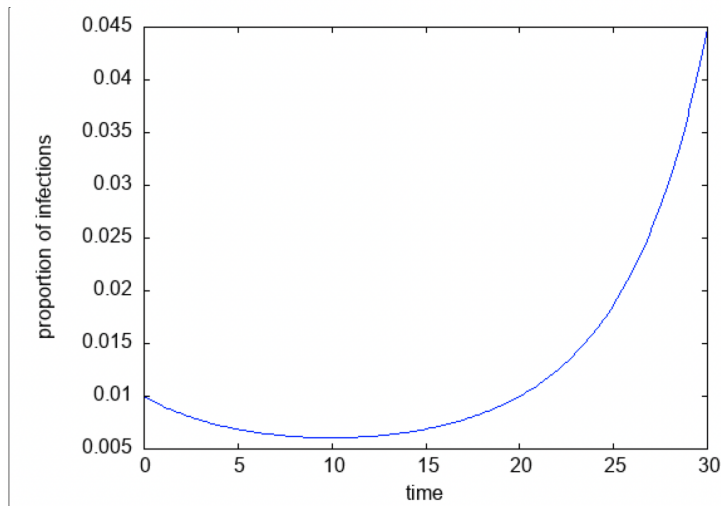
(i) Based on your general solution, you use maxima to plot the proportion of the population who is infected, starting with 1% infected individuals and no susceptible individuals:

```
wxplot2d([I0·%e^(-t·κ+(t^2·β·δ)/2+S0·t·β\-t·α)],[t, 0, 30], [xlabel, "time"],
    [ylabel, "proportion of infections"]), S0:0, I0:0.01, κ:0.1, β:0.2, δ:0.05, α:0.01;
```


Circle one (**True or False**) for each of the following statements based on the above results and plot. Assume no other variable or parameter is changed besides the one discussed.

- **True** or **False:** The proportion of infected individuals would decline regardless of the initial proportion of infected individuals, as long as there are some ($I_0 > 0$). Because there are no susceptibles

- **True** or **False:** The proportion of infected individuals would decline regardless of the initial proportion of susceptible individuals, as long as there are some ($S_0 > 0$). If there were enough susceptibles initially, then infections could rise initially

- **True** or **False:** The reason that the proportion of infected individuals declines initially is because there are too few infected individuals at first in the population (I_0 is too small). There are too few susceptibles

- **True** or **False:** The reason that the proportion of infected individuals eventually rises is because enough individuals have lost immunity (moving from the resistant $R(t)$ class to the susceptible class $S(t)$), so that the disease has a positive growth rate ($r > 0$, as defined in parts (a) and (b)).